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by

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Abstract

Pricing and hedging of long-term interest rate sensitive products require to extrapolate the term structure beyond observable maturities. For the resulting limiting term structure we show two results by postulating no arbitrage in a bond market with infinitely increasing maturities: long zero-bond yields and long forward rates (i) are monotonically increasing and (ii) equal their minimal future value. Both results constrain the asymptotic maturity behavior of stochastic yield curves. They are fairly general and extend beyond semimartingale modeling. Hence our framework embeds arbitrage-free term structure models and imposes restrictions on their specification.

Keywords: bond markets, yield curve, long forward rates, no arbitrage, asymptotic maturity
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1 Introduction

A central issue in mathematical finance is the stochastic modeling of yield curves. It is popular to set up bond market models on the term structure of interest rates. The literature has advanced in modeling the evolution of the term structure dynamically by integrating asset-pricing theory. Traditionally, many models in this approach concentrate on the short-term behavior. However, the focus on the long rate is also of great interest: The long-term behavior is essential for pricing and hedging of long-term interest rate sensitive products. These products include fixed-income securities, insurance and annuity contracts, and perpetuities. For these instruments finance practitioners need to extrapolate the term structure beyond limited observable maturities. Hence models are required, which capture the evolution of the yield curve for longer maturities. To derive joint properties of such models, we examine the limiting term structure in general and show two substantial results. Under no arbitrage in a frictionless bond market with infinitely increasing maturities, long zero-bond yields and long forward rates satisfy two properties:

(i) *Asymptotic Monotonicity:* Both rates are monotonically increasing in time,

(ii) *Asymptotic Minimality:* Both rates equal their minimal future value.

This article derives both properties in a general framework for term structure models.

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The first result of asymptotic monotonicity states that both asymptotic rates cannot fall over time and hence excludes that tomorrow's long rate is less than today's long rate. Both rates still may increase, but they cannot increase almost surely, which is denied by the second result of asymptotic minimality. It excludes systematic jumps of the long rates. Hence the range of possible realizations of tomorrow's long yield is not bounded away from today's long yield. Consequently both results cause that asymptotic maturity behavior of the term structure is not arbitrary. They reduce potential realizations of stochastic yields curves by excluding a multitude of asymptotic behavior under no arbitrage.

Both results are fairly general, since we derive them under weak assumptions. To show properties of the long rates, we assume their existence. The only further assumption is to postulate no arbitrage in a bond market with infinitely increasing maturities. Instead of addressing a certain term structure model, we rather present a framework for term structure models. This framework refers to a frictionless bond market with continuous trading and is based on stochastic calculus. Focusing on a continuous-time setting is not essential, since the results can be derived analogously in a discrete-time setting. A model in our framework is given by a family of bond prices for any maturity. Opposed to most approaches to bond markets, we do not restrict these prices to a certain class of processes. As a result of this generality, our framework embeds virtually all existing arbitrage-free term structure models. In consequence the results and their implications apply to all these models. Our framework is even open to bond prices, which are not modeled as semimartingales, for example the fractional Brownian motion. Allowing for non-semimartingales is significant, since non-semimartingales appear more regularly in the empirical literature estimating price processes, see [18] and references therein. Jarrow, Protter and Sayit [14] recently showed that non-semimartingales do not necessarily impose arbitrage possibilities.

Summing up, asymptotic monotonicity and minimality are important, as they exclude various behavior of limiting yield curves. Since we derive them in a general framework, they impose severe restrictions on the long-term behavior of virtually all term structure models. These theoretical implications serve as a benchmark for modelers specifying an arbitrage-free term structure. Specifically, setting up the asymptotic yield or forward rate as a diffusion process or a process with systematic jumps necessarily imposes arbitrage opportunities.

The actual asymptotic behavior of 16 well-known term structure models is analyzed in detail by Yao [25]. All these models satisfy asymptotic monotonicity, although they are not necessarily arbitrage-free. In the models of Dothan [7] and Heath, Jarrow and Morton [11] the result applies under existence of the long rates without further parameter specification. On first sight the result seems to be violated in the model by Brennan and Schwartz [3]. The long rate is specified exogenously and decreases under certain parameter choice, but it refers to a consol bond, instead of a zero-coupon bond. In some term structure models, including Vasiček [23], Cox, Ingersoll and Ross [5] and Chen [4], long bond yields are constant over time. This specific behavior suggests to consider the generalization of our two results, stating that long yields are constant over time. However, it is impossible to derive this generalization, which in consequence closes our results. This can be seen by considering the discrete binomial model by Ho and Lee [12]. This model is an example of an arbitrage-free bond market with infinitely increasing maturities and its long yield rises with positive Bernoulli-probability at each discrete time point. Considering this model is furthermore illustrative for both results, since they are not violated, although the bond yield underlies permanent downward shocks. For this discussion we refer to [8].

The first result of asymptotic monotonicity is initially addressed by Dybvig, Ingersoll and Ross [8]. They came up with the genuine idea of showing this result by a no arbitrage-argument. Since it was also addressed by several other authors, we compare the different approaches and comment on our respective generalizations. To our best knowledge Dybvig, Ingersoll and Ross [8] are the only authors to address the second result of asymptotic minimality. They derive it in finite sample spaces, but remarkably they provide a counter-example for infinite spaces. The resulting contradiction is clarified by a detailed analysis of alternative definitions of arbitrage. The counter-example works due to a slightly inconsistent choice of convergence criteria for long yields and arbitrage.

This paper is organized as follows: The following Section provides the general framework for term structure models. In this setting Section 3 presents the proof of asymptotic monotonicity by explicitly constructing an arbitrage strategy. Using a related proof, asymptotic minimality is established in Section 4. The related literature and the apparent contradiction are discussed in Section 5. Section 6 concludes.

2 Modeling the Bond Market and Arbitrage

This section presents the formal setting of the bond market and introduces the notion of arbitrage in the limit. The bond market is defined by a family of zero-coupon bond prices, confer also [21]. A *zero-coupon bond* is a financial security that pays one unit of cash to its holder at a fixed later date T, called *maturity*. We assume these bonds to be default-free.

Definition 1 (Bond Market) *A* Bond Market *consists of a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *with filtration* $(\mathcal{F}_t)_{t\geq 0}$ *and an adapted locally bounded process* $B(\cdot, T)$ *for every* $T \geq 0$.

This definition is very general, since the only two restrictions for bond prices, adaption and local boundedness, are not very demanding.[1] Bond prices are just assumed to be given, so they can be observed in some real market or generated by some procedure. Whereas bond prices are traditionally modeled by local semimartingales, which include diffusions and càdlàg jump processes, our approach is not limited to semimartingales. Depending on the properties of the bond prices the class of trading strategies will be restricted to be integrable with respect to these bonds. For arbitrary bond prices processes the integrability is achieved for simple integrands (for more details see below). Hence a model in our general approach is given by a family of almost arbitrary bond prices. The class of bond prices then determines the set of admissible strategies and the severity of the assumption of no arbitrage. More demanding is requiring the existence of bond price processes for arbitrarily large T. Notice that we do not need the existence of the limiting process $\lim_{T\to\infty} B(\cdot, T)$, which in practice equals zero. As usual we assume the filtration to satisfy the usual conditions.

For a given bond we can consider the constant yield from holding it over the time interval $[t, T]$. This yield-to-maturity is called *bond yield* or *zero-coupon rate*, denoted by $z(t, T)$ and defined via

$$B(t, T) = \exp\left(-z(t, T)(T - t)\right). \tag{1}$$

[1] The local boundedness is formally given by the existence of a sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times, increasing almost surely to infinity, such that the stopped processes $B(t \wedge \tau_n, T)$ are almost surely uniformly bounded over time for all $n \in \mathbb{N}$. This restriction does not seem to be severe, since in practice bond prices lie between zero and one.

The *instantaneous forward rate* is the interest rate fixed at time t for lending over the infinitesimal interval $[T, T + dt]$. It is denoted by $f(t, T)$ and connected to bond prices and bond yields by

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right),$$
$$z(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du. \qquad (2)$$

The concept of continuously compounded forward rates is a mathematical idealization, whereas bond prices and bond yields are observable in practice. Nevertheless all three concepts are, from a theoretical point of view, equivalent in defining the so-called *term structure of interest rates* or *yield curve* at time t, which relates maturity T to the bond yield $z(t,T)$. For our asymptotic view of the term structure we define the following almost sure limits

$$\text{the } \textit{long bond yield} \quad z_L(t) := \lim_{T\to\infty} z(t,T), \qquad (3)$$
$$\text{and the } \textit{long forward rate} \quad f_L(t) := \lim_{T\to\infty} f(t,T),$$

which do not necessarily exist. An argument for the existence is given by Yao [25] and empirically by Malkiel [20], who show that the yield curve levels out for growing maturity.

We now consider how bonds can be traded in our market. Therefore we assume that an investor trades a finite number of bonds, say k, out of the infinite number of bonds available. Thus let $\mathcal{T} = (T^1, ..., T^k)$ be a vector of maturities. Infinite portfolios, which we do not need to derive our result, are considered in the rigorous generalization by Björk et al. [2]. Moreover, an investor can invest into the numéraire, in which the bonds are expressed.[2] A *bond trading strategy* is a pair $(\Phi, \mathcal{T})$, where $\Phi = (\Phi^0, ..., \Phi^k)$ is a predictable real-valued process on $(\Omega, \mathcal{F}, \mathbf{P})$. $\Phi^0(t)$ denotes the units of the numéraire and $\Phi^i(t)$ denotes the units of the bond $B(\cdot, T^i)$, which are held at time t. Since the numéraire is a traded asset and its price process, priced by itself, equals one at all times, it serves as a cash box, which finances buys and sells. Hence we do not have to check, if bond trading strategies are self-financing. Notice that our results do not depend on the choice of numéraire.

Definition 2 (Admissibility) *A bond trading strategy* $(\Phi, \mathcal{T})$ *with* $\Phi(0) = \mathbf{0}$ *is called* admissible*, if the Itō-integral,*[3] *denoted by*

$$\big(\Phi \circ B(\cdot, \mathcal{T})\big) = \Big(\big(\Phi \circ B(\cdot, \mathcal{T})\big)_t\Big)_{t\geq 0},$$

is well-defined and uniformly bounded from below.[4]

Hence the class of admissible strategies depends crucially on the class of given bond prices. Simplifying one can state that the bigger the class of integrators given by the model in question, the

[2] To consider discounted bond prices, we can change the numéraire e.g. to the money market account, given by $B(t) := \exp(\int_0^t f(u,u)du)$, or to an account rolling over certain bonds, e.g. $B(t) := \frac{B(t,[t]+1)}{\prod_{n=1}^{[t]+1} B(n-1,n)}$.

[3] Our approach does not depend on the stochastic integration theory of the Itō-integral. Our proofs only require one-dimensional simple integrands and thus work in a general setting with any stochastic integral, which equals the natural integral on the class of simple integrands. A reasonable restriction for the integral would be linearity and non-anticipation to exclude trivial arbitrage strategies.

[4] The boundedness of the integral is required to exclude trivial arbitrage by doubling strategies and formally given by the existence of some $K \in \mathbb{R}$ with $\big(\Phi^i \circ B(\cdot, T^i)\big)_t \geq K$ almost surely for all i and $t \geq 0$.

smaller the corresponding class of admissible integrands. If the bond prices are for example continuous, square-integrable martingales, the Itō integral is well-defined for progressively measurable strategies, which are square-integrable w.r.t. to the Doléan-measure, see e.g. [16], Chapter 3.2. For arbitrary bond price processes of Definition 1, simple integrands are fitting, which ensures the generality of our approach. A *simple integrand* with bounded support is a sum of the form $\Phi = \sum_{l=1}^{m} f_l \cdot \mathbf{1}_{(\tau_{l-1},\tau_l]}$, where $0 \leq \tau_1 \leq ... \leq \tau_m$ are finite stopping times and the functions f_l are $\mathcal{F}_{\tau_{l-1}}$-measurable. In this case the natural stochastic integral, defined by the following Riemann-sums

$$\left(\Phi \circ B\right)_t = \sum_{l=1}^{m} f_l \cdot \left(B(t \wedge \tau_l) - B(t \wedge \tau_{l-1})\right),$$

is well-defined for any adapted process B. A simple integrand with uniformly bounded functions $f_1, ..., f_m$ is hence admissible for any bond price of Definition 1, since these bond prices are locally bounded and the stopping times $\tau_1, ..., \tau_m$ are finite. So considering non-semimartingales poses no technical problem.

The last concept we formalize is the concept of arbitrage. Intuitively, an arbitrage is described by a risk-free strategy with a chance of "making something out of nothing". Instead of referring to value of the strategy at date t, given by the skalar product with the bond prices, we directly refer to its stochastic integral, which accumulates gains and losses up to time t. An admissible bond trading strategy $(\Phi, \mathcal{T})$ is called an *arbitrage* if there exists a point in time t with $t \leq \min_i T^i$ and

$$\begin{aligned}
&\left(\Phi \circ B(\cdot, \mathcal{T})\right)_t \geq 0,\\
&\mathbf{P}\left(\left(\Phi \circ B(\cdot, \mathcal{T})\right)_t > 0\right) > 0.
\end{aligned}$$

As we interested in the behavior of limiting yields, we need to define an arbitrage trading bonds, which approximate the asymptotic bond. This is done by considering a sequence of bond trading strategies, in which at least one maturity increases infinitely. Not all maturities are required to explode, since it still should be possible to invest into finite bonds.

Definition 3 (Arbitrage in the Limit) *A sequence of admissible bond trading strategies* $(\Phi_j, \mathcal{T}_j)_{j\in\mathbb{N}}$ *with* $\lim_{j\to\infty} T_j^i = \infty$ *for some* i *is called an* arbitrage in the limit *if there exists a point in time* t *with* $t \leq \min_i \lim_{j\to\infty} T_j^i$ *and*

$$\begin{aligned}
&\lim_{j\to\infty} \left(\Phi_j \circ B(\cdot, \mathcal{T}_j)\right)_t \geq 0,\\
&\mathbf{P}\left(\lim_{j\to\infty} \left(\Phi_j \circ B(\cdot, \mathcal{T}_j)\right)_t > 0\right) > 0.
\end{aligned}$$

The limit in this definition is an almost sure limit. Thus the integral has to converge almost surely against some random variable, which is exceptionally allowed to take infinite values with positive probability. Hence convergence of this integral does not imply its boundedness. Note that the maturities of traded bonds explode and not the trading time. The absence of arbitrage in the limit in the bond market is denoted by (NAL). We close the section by comparing our definitions to classical definitions of arbitrage. Our first definition of arbitrage is just the standard concept from the literature, see e.g. Harrison and Pliska [10]. Our second definition of arbitrage in the limit involves a sequence of trading strategies. A limiting procedure can also be found in the prominent no-arbitrage concepts *no free lunch* by Kreps [17], *no free lunch with vanishing risk* and *no free lunch with bounded risk* by Delbaen and Schachermayer [6]. All four concepts have in common that a sequence of admissible strategies is considered, in which arbitrage is realized in the limit with respect to a certain topology. But there are also substantial differences. The no free lunch concepts consider a stock price process and the integrator is fixed throughout the sequence. In our concept the integrator is a bond price and it varies within the sequence, since at least one maturity explodes. The reason to introduce the limiting procedure differs also. Whereas the absence of arbitrage in sense of Harrison and Pliska [10] is too weak to imply the existence of an equivalent martingale measure in infinite spaces, the no free lunch concepts are strong enough for this implication due to the passage to the limit. In our approach the reason is to incorporate long bond yields, which are approximated by a sequence of bonds with infinitely increasing maturities. To our knowledge, the notion, which is closest to (NAL), is given by Föllmer and Schachermayer [9]. They interpret the definition of *strong asymptotic arbitrage*, which is introduced by Kabanov and Kramkov [15], for a varying time horizon. It differs from (NAL) as an arbitrage has to be arbitrarily big with arbitrarily small probability. Since the arbitrage strategies constructed in our proofs also satisfy this notion, if it is transferred to bond markets, our results also hold true under assuming the absence of strong asymptotic arbitrage.

3 Asymptotic Monotonicity

In this section we prove that long bond yields can never fall over time in an arbitrage-free setting. Given the bond market from the previous section the following holds:

Theorem 4 (Asymptotic Monotonicity) *If $z_L(s)$ and $z_L(t)$ exist almost surely for $s < t$, then under (NAL) it holds*

$$z_L(s) \leq z_L(t) \quad a.s.$$

Before proving the result we give some intuition: The theorem states that tomorrow's infinite bond yield can never be less than today's infinite bond yield. If this statement was wrong, the long bond yield would fall with positive probability. Then we would buy a bond today with high yield, and this buy would be relatively cheap by relation (1), and sell a bond of the same maturity with a potentially lower yield tomorrow. At maturity, which grows to infinity, the bonds are neutralizing each other. We buy such a small amount that today's costs are asymptotically zero. In case the long bond yield falls, tomorrow's bond is more expensive and we have a profit. This strategy yields an arbitrage in the limit contradicting the no arbitrage-assumption. Note that the gain is realized in finite time t, only the maturity of traded bonds tends to infinity.

Proof. We start showing asymptotic monotonicity by conditioning on $z_L(s)$, which exists almost surely by assumption. Hence we define the conditional probability measure $\mathbf{P}_{z_L(s)} := \mathbf{P}(\,\cdot\,|z_L(s))$. To prove in the first step

$$z_L(s) \leq z_L(t) \quad \mathbf{P}_{z_L(s)}\text{-a.s.}$$

we assume the contrary $\mathbf{P}_{z_L(s)}(z_L(s) \leq z_L(t)) < 1$, which is equivalent to $\mathbf{P}_{z_L(s)}(z_L(s) > z_L(t)) > 0$ and, thus it follows

$$\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big) < z_L(s)$$

by definition of $\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big) := \sup_{z\in\mathbb{R}}\big\{\mathbf{P}_{z_L(s)}(z_L(t) < z) = 0\big\}$. For details of this definition we refer to footnote.[5] We set $y := \frac{1}{2}\,z_L(s) + \frac{1}{2}\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)$ and receive a

[5] The essential infimum of a random variable X is well-known as its minimum up to null-sets, defined by the number $\operatorname{ess\,inf}(X) := \sup_{z\in\mathbb{R}}\{\,\mathbf{P}(X < z) = 0\,\}$. In our case we need to condition the essential infimum on $z_L(s)$.

random variable with

$$\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big) < y < z_L(s).$$

Then, the set

$$\Omega_1 := \{z_L(t) < y\}$$

has a positive probability under $\mathbf{P}_{z_L(s)}$. To derive an arbitrage in the limit we consider the following sequence of one-dimensional bond trading strategies $(\Phi_j, \mathcal{T}_j)_{j\in\mathbb{N}}$, given by

$$\Phi_j(u) := \exp(y(j-s))\ \mathbf{1}_{(s,t]}(u), \quad \mathcal{T}_j := j.$$

Each strategy of the sequence is a trivial buy and hold strategy: at date s we buy $\exp(y(j-s))$ units of the bond with maturity j and sell them at date t. We first show that each strategy is predictable, a property which is questioned in [13]. Since the strategies are constant up to a jump after date s, it suffices to show that $\Phi(s+)$ is $\mathcal{F}_s$-measurable. Its measurability crucially depends on the random variable y. This y is $\mathcal{F}_s$-measurable, since it is the sum of two $\mathcal{F}_s$-measurable random variables: $z_L(s)$ and the essential infimum. Note that the essential infimum does not depend on $z_L(t)(\omega)$, which is realized at time t. It is rather a property of the measure $\mathbf{P}_{z_L(s)}$, which is known at time s. Second, we derive that the strategies are admissible: Each strategy is a simple integrand by construction. Hence the stochastic integral is well-defined for any bond price processes in the natural way by

$$\begin{aligned}\big(\Phi_j \circ B(\cdot,\mathcal{T}_j)\big)_u :=& \exp(y(j-s))\,\big(B(\min(u,t),\mathcal{T}_j) - B(\min(u,s),\mathcal{T}_j)\big)\\ =& \exp(y(j-s))\,\big(B(u,\mathcal{T}_j) - B(s,\mathcal{T}_j)\big)\ \mathbf{1}_{(s,t]}(u)\\ &+ \exp(y(j-s))\,\big(B(t,\mathcal{T}_j) - B(s,\mathcal{T}_j)\big)\ \mathbf{1}_{(t,\infty)}(u).\end{aligned}$$

For any given j the number of shares $\exp(y(j-s))$ is uniformly bounded by choice of y and the convergence of $z_L(s)$. As argued in the previous section, this integral thus yields the admissibility of the given strategies for arbitrary bond prices of Definition 1. Third, we show that the sequence constitutes an arbitrage in the limit and consider the integral at time t, which equals by relation (1) for all j

$$\begin{aligned}\big(\Phi_j \circ B(\cdot,\mathcal{T}_j)\big)_t =& \exp(y(j-s))\,\big(B(t,j) - B(s,j)\big)\\ =& \exp(y(j-s))\ \exp(-z(t,j)(j-t))\\ &- \exp(y(j-s))\ \exp(-z(s,j)(j-s))\\ \geq& -\exp((y-z(s,j))(j-s)).\end{aligned}$$

The almost sure existence of $\lim_{j\to\infty}\big(\Phi_j \circ B(\cdot,\mathcal{T}_j)\big)_t$ is ensured by the almost sure convergence of $z_L(s)$ and $z_L(t)$. As j grows to infinity, the last term converges almost surely to zero, since $y < z_L(s)$ holds almost surely. This yields

$$\lim_{j\to\infty}\big(\Phi_j \circ B(\cdot,\mathcal{T}_j)\big)_t \geq 0.$$

Hence we consider the following generalization: Let X, Y be two random variables. We define the essential infimum with respect to a measure conditioned on Y by setting

$$\operatorname{ess\,inf}\big(X\,\big|\,Y\big) := \sup_{z\in\mathbb{R}}\{\,\mathbf{P}(X < z\,|\,Y) = 0\,\}$$

and $\operatorname{ess\,inf}\big(X\,\big|\,Y\big) := -\infty$, if the set $\{z \in \mathbb{R} \mid \mathbf{P}(X < z\,|\,Y) = 0\}$ is empty. Notice that by taking the supremum over this markov kernel, its measure property is eliminated and the measurability in its second argument remains. Hence $\operatorname{ess\,inf}\big(X\,\big|\,Y\big)$ is a $\sigma(Y)$-measurable random variable mapping onto $\mathbb{R}\cup\{-\infty\}$. For a rigorous and extensive discussion of conditional essential suprema we refer to [1].

On the set Ω_1 the integral even grows unboundedly, since the first summand explodes almost surely on Ω_1. Since Ω_1 has a positive probability, it follows

$$\mathbf{P}_{z_L(s)}\big(\lim_{j\to\infty}\big(\Phi_j \circ B(\cdot,\mathcal{T}_j)\big)_t > 0\big) > 0.$$

Since the conditional measure $\mathbf{P}_{z_L(s)}$ is equivalent to $\mathbf{P}$, the sequence $(\Phi_j, \mathcal{T}_j)_{j\in\mathbb{N}}$ constitutes is an arbitrage in the limit. This contradicts the assumption of (NAL) and we have

$$\mathbf{P}_{z_L(s)}(z_L(s) \le z_L(t)) = 1.$$

We have proved the first step and now omit conditioning on $z_L(s)$. Writing the conditional probability as conditional expectation it follows almost surely

$$\begin{aligned} 1 &= \int_\Omega \mathbf{P}(z_L(s) \le z_L(t) \,|\, z_L(s))\, \mathbf{P}(d\omega) \\ &= \mathbf{E}_\mathbf{P}\Big[\mathbf{E}\big(\mathbf{1}_{\{z_L(s)\le z_L(t)\}} \,\big|\, z_L(s)\big)\Big] \\ &= \mathbf{E}_\mathbf{P}\Big[\mathbf{1}_{\{z_L(s)\le z_L(t)\}}\Big], \end{aligned}$$

which completes the proof. ∎

As shown in [8], Theorem 1 for a discrete time setting or easily derived by relation (2), $f_L(t)$ equals $z_L(t)$ almost surely, if $f_L(t)$ exists almost surely. As a result, the following corollary states that the infinite forward rate cannot fall over time.

Corollary 5 (Asymptotic Monotonicity) *If $f_L(s)$ and $f_L(t)$ exist almost surely for $s < t$, then under (NAL) it holds*

$$f_L(s) \le f_L(t) \quad a.s.$$

4 Asymptotic Minimality

In the previous section we have seen that the long bond yield cannot fall over time. It certainly may rise, what can be seen e.g. by considering the discrete binomial model by Ho and Lee [12], whose long yield rises with positive Bernoulli-probability at each discrete time point. But may it rise almost surely? This question is denied by the following result: the long bond yield always equals its minimal future value. Thus today's long bond yield cannot be bounded away from the support of possible future long yields and so jumps with probability one are excluded. The adequate notion of minimum for our purpose is given in Footnote 5 in the previous Section. In the light of this notion asymptotic minimality states that every neighborhood of $z_L(s)$ has a positive probability in the distribution of $z_L(t)$ conditioned on $z_L(s)$.

Theorem 6 (Asymptotic Minimality) *If $z_L(s)$ and $z_L(t)$ exist almost surely for $s < t$, then under (NAL) it holds*

$$z_L(s) = \operatorname{ess\,inf}\big(z_L(t) \,\big|\, z_L(s)\big) \quad a.s.$$

Before proving the result we again give some intuition: If the theorem was wrong, tomorrow's long bond yield would rise with probability one due to asymptotic monotonicity. Then we could sell an expensive bond with low yield today and buy a cheaper bond of the same maturity with a definitely higher yield tomorrow. At maturity, which grows to infinity, the bonds are neutralizing each other. We sell a precise number of shares, such that we are paid today to enter the position and have asymptotically no costs tomorrow. This arbitrage in the limit is realized in finite time and contradicts the assumption.

Proof. This proof is a condensed version, since its structure is similar to the proof of asymptotic monotonicity, Theorem 4. To show asymptotic minimality we again condition on $z_L(s)$ and set $\mathbf{P}_{z_L(s)} := \mathbf{P}(\,\cdot\,|z_L(s))$. The following inequalities hold $\mathbf{P}_{z_L(s)}$-almost surely. By asymptotic monotonicity we already know, that

$$z_L(s) \leq \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)$$

holds. To show the remaining inequality

$$z_L(s) \geq \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big), \tag{4}$$

we assume the contrary $z_L(s) < \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)$ and set $\bar{y} := \frac{1}{2}\,z_L(s) + \frac{1}{2}\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)$ with

$$z_L(s) < \bar{y} < \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big).$$

To derive an arbitrage in the limit we consider the following sequence of one-dimensional buy and hold strategies, which sell depending on $\bar{y}$ a number of the bond with maturity j at date s to rebuy it at date t:

$$\Psi_j(u) := -\exp(\bar{y}(j-s))\;\mathbf{1}_{(s,t]}(u), \quad \mathcal{T}_j := j.$$

The measurability of the processes Ψ_j again crucially depends on the random variable $\bar{y}$, which is the sum of $z_L(s)$ and the essential infimum, which both are known at time s. Thus $\Psi_j(s+)$ is $\mathcal{F}_s$-measurable and all strategies are predictable. Each strategy is a simple integrand and thus the stochastic integral is well-defined in the natural way. This integral is uniformly bounded from below for any fixed j by choice of $\bar{y}$, and hence yields the admissibility of each strategy for all bond prices. To see that this sequence of admissible bond trading strategies constitutes an arbitrage in the limit, we consider the integral at date t, which equals by relation (1)

$$\begin{aligned}\big(\Psi_j \circ B(\cdot,\mathcal{T}_j)\big)_t &= -\exp(\bar{y}(j-s))\,\big(B(t,\mathcal{T}_j) - B(s,\mathcal{T}_j)\big)\\ &= -\exp(\bar{y}(j-s))\,\exp(-z(t,j)(j-t))\\ &\quad + \exp(\bar{y}(j-s))\,\exp(-z(s,j)(j-s)).\end{aligned}$$

As j grows to infinity, the first summand converges to zero and the second summand explodes, since $z_L(s) < \bar{y} < z_L(t)$ holds $\mathbf{P}_{z_L(s)}$-almost surely. This yields

$$\lim_{j\to\infty}\big(\Psi_j \circ B(\cdot,\mathcal{T}_j)\big)_t \geq 0.$$

and furthermore

$$\mathbf{P}_{z_L(s)}\Big(\lim_{j\to\infty}\big(\Psi_j \circ B(\cdot,\mathcal{T}_j)\big)_t > 0\Big) = 1.$$

Consequently the sequence $(\Psi_j, \mathcal{T}_j)_{j\in\mathbb{N}}$ is an arbitrage in the limit. This contradicts (NAL) and we have shown the remaining inequality (4), which completes the proof. ■

The result of asymptotic minimality can again be expanded to long forward rates.

Corollary 7 (Asymptotic Minimality) *If $f_L(s)$ and $f_L(t)$ exist almost surely for $s < t$, then under (NAL) it holds*

$$f_L(s) = \operatorname{ess\,inf}\big(f_L(t)\,\big|\,f_L(s)\big) \quad a.s.$$

5 The Literature

In the first part of this section we compare the literature on asymptotic monotonicity to our approach. Since several authors worked on this topic, we concentrate on a few prominent papers, which are closely related to ours. In the second part of the section we refer to asymptotic minimality and clarify a contradiction to existing literature.

5.1 Asymptotic Monotonicity

The most important contribution to asymptotic monotonicity is by Dybvig, Ingersoll and Ross [8]. They came up with the genuine idea of showing this result by a no arbitrage-argument and initiated the proceeding literature. Since their paper is formulated in an intuitive way, we make their original idea more rigorous in this paper. For this purpose, we provide a stringent formal setting, in which all objects are unambiguously defined in mathematical terms. Compared to [8], this setting is extended in several aspects. We stress out two aspects: First by using a continuous-time setting we allow for continuous trading, and second we expand the stochastic modeling from one point in time to the continuum $[\,0,\infty)$.

McCulloch [19] comments on [8] and states that the proof is defective, since it includes an error. This critic is valid, but it only refers to the proof in the body of [8]. The proof in the Appendix of [8] is not affected, since the problem stems from the set $\{y = \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)\}$, where the invested amount equals the essential infimum, and y is chosen to be strictly greater than the infimum. Neither our proof is affected for the same reason.

Yao [24] derives asymptotic monotonicity rigorously under additional assumptions in a jump-diffusion context.

Hubalek, Klein and Teichmann [13] provide a stringent proof of asymptotic monotonicity, which is quite different from ours. Whereas in their setting no arbitrage is given by the existence of an equivalent martingale measure, we approach arbitrage in a less abstract way by a positive definition. We see two advantages in this approach: First by introducing the concrete notion of arbitrage, we can construct the arbitrage strategy explicitly and tell an arbitrageur to buy how many of which bonds. Hence our proof is more illustrative. Second we do not require the fundamental theorem of asset pricing and, thus we are able to leave the common semimartingale-setting. As a result, whereas in the setting of [13] bond prices are modeled as semimartingales, our setting is extended to a broader class, including non-semimartingales.

Without affecting the validity of their elegant and conveniently brief proof, Hubalek, Klein and Teichmann [13] criticize [8] erroneously in proposing that the strategy is anticipative. They state that in consequence one has to assume implicitly in [8] that the long bond yield at time t, denoted by $z_L(t)$, is $\mathcal{F}_s$-measurable for $s < t$. But this assumption is not necessary, since the strategy in [8] is not anticipating: The strategy does not depend on $z_L(t)$, but on its essential infimum. The essential infimum is in turn a property of the distribution, which does not depend on the realization of $z_L(t)$. In the setting of [8], with stochastic modeling only at date t, this infimum is just a number and its measurability poses no problem. In our general setting it neither poses a problem. By considering the essential infimum conditioned on $z_L(s)$, which is hence known at date s, we can construct an arbitrage strategy, which is not anticipating.

5.2 Asymptotic Minimality

To our best knowledge Dybvig, Ingersoll and Ross [8] are the only authors to address the second result of asymptotic minimality. They derive the result in spaces with a finite number of states. But remarkably they provide a counter-example for infinite spaces. Thus we are most likely the first to claim asymptotic minimality in infinite spaces. In this section we clarify the resulting contradiction in detail and we show that it is only ostensible. The reason for the contradiction does not lie in the different setting of the bond market mentioned above. It is solely located in the definition of arbitrage. The notion of arbitrage has a great impact on the asymptotic maturity behavior. Hence we start an analysis of alternative definitions of arbitrage, which finally shows that the counter-example works due to a slightly inconsistent choice of convergence criteria for long yields and arbitrage.

We quote the definition of arbitrage in [8]: *An arbitrage opportunity is a sequence of net trades (allowing free disposal), such that either (i) the price tends to zero but the payoff tends uniformly to a nonnegative random variable that is positive with positive probability or (ii) the price tends to a negative number (you are paid to enter the position) but the payoff tends uniformly to a nonnegative random variable.*

This intuitive definition is open to many formal interpretations, which include our Definition 3. But there are also two qualitative differences: the claim for uniform convergence and the addition "allowing free disposal". In order to formalize this definition mathematically we have to define the price and the payoff of an arbitrage. This purpose is not as clear as it seems, and there are several ways: e.g. setting the price equal to (i) the negative part of the integral, (ii) the value at a certain date or (iii) the sum of the values of traded bonds at their entering times. The purpose is easier for one-dimensional buy-and-hold-strategies. The following intuitive definition helps to understand the setting of [8] formally: "price" is defined as value of the arbitrage at the time s, when we enter the position, and "payoff" as the value at later trading time t, where the value at time u is defined by $\Phi(u) \cdot B(u, \mathcal{T})$. This definition is consistent with [8] in the sense, that in their setting this price is deterministic and this payoff is stochastic, and it makes sense to claim ω-uniform convergence. Notice that in the setting of [8] date t is the only stochastically modeled point in time. If we formalize the definition of [8] for a general probability space, in which date s is also stochastic, we have to choose a convergence criterium for the price. We analyze two obvious criteria: (i) almost sure convergence and (ii) almost sure uniform convergence.

- In the first case we define arbitrage in the limit according to Definition 3, and require additionally that the payoff has to converge almost sure uniformly. With this notion of arbitrage we replicate the results of [8]. We can prove asymptotic monotonicity, but it requires a more extensive proof and the introduction of free disposal. By free disposal, which intuitively means throwing away money, we are able to truncate the exploding payoff and hence achieve uniform convergence. In this setting counter-example in [8] of a strictly growing bond yield in an infinite sample space is valid and can also be transferred to a continuous-time setting. Hence asymptotic minimality does not hold in this case. Consequently, by the discussed notion of arbitrage and the general setting of section 2 we provide the formal basis for the reasoning in [8].

- We now study the second case of transferring the definition of arbitrage in [8] to a general probability space, in which we again refer to Definition 3 and additionally claim for both, price and payoff, almost sure uniform convergence. In this setting it is easy to construct an example, in which the long bond yield strictly falls. The trick is to consider a bond yield, which converges almost surely, but not uniformly at time s. If we try to construct an arbitrage contradicting the falling yield, the arbitrageur has to trade at date s and, hence, the price, which necessarily depends on the bond yield at time s, will not converge uniformly to zero. In this case, free disposal cannot help out, since we cannot truncate the price without losing its almost-sure convergence to zero. Thus we cannot construct an arbitrage with a uniformly convergent price and asymptotic monotonicity does not hold in this case. Furthermore asymptotic minimality does not hold either, since every arbitrage in this case is also an arbitrage in the first case, in which asymptotic minimality fails to hold.

Summing up, we state that the definition of arbitrage in the limit has great impact on the behavior of long bond yields and long forward rates. As the trick of the counter-examples in both cases is a bond yield which converges almost surely, but not uniformly, it is self-evident to consider bond yields, which converge uniformly. Indeed, if the limit of bond yield (3) is defined as an almost sure uniform limit, asymptotic monotonicity and minimality hold for all three considered types of arbitrage in the limit. The proofs are analogous to those presented and the uniform convergence of price and payoff is ensured by the uniform convergence of the bond yield.

Finally we discuss which definition seems to make most sense. As we analyze the asymptotics of bond yields it is necessary to define the non-standard limit of arbitrage. Claiming for uniform convergence seems inappropriate in this context. Under uniform convergence long bond yields behave very differently. Concerning asymptotic monotonicity and minimality the behavior is asymmetric and appears arbitrary, since in the first discussed case minimality fails to hold and in the second case minimality and monotonicity break down. This absurd finding is an argument against definitions of arbitrage transferred from the setting of [8]. In addition, the crucial reason of this mutated behavior is trivial and artificial: the convergence criterium in the definition of the long bond yield. It is not specified in [8], but it has to be an almost sure limit to make their counter-example valid. If one is consequent and defines the long bond yield as a uniform limit the ordinary behavior is reestablished regardless to the type of analyzed definition of arbitrage. These findings advise to choose convergence criteria of bond yield and arbitrage consistently, and thus consistent definitions of arbitrage in the limit make most sense. We conclude, that with any consistent choice of convergence criteria, asymptotic monotonicity and minimality of bond yield and forward rate hold in an arbitrage-free setting.

6 Conclusion

In this article we approach bond markets by considering an arbitrage-free family of bond prices with infinitely increasing maturities, which is not limited to the class of semi-martingales. In this general setting we derive the Dybvig-Ingersoll-Ross result of non-falling long bond yields. Our proof is based on an explicitly constructed arbitrage strategy, which is not anticipating, as proposed in the literature. Furthermore we derive a second asymptotic result: Long bond yields and forward rates equal their minimal future value. For this finding [8] provides a counter-example and the reason for the apparent contradiction is analyzed in detail. This analysis concludes that the second result holds in any consistent arbitrage-free setting. Both results impose restrictions on arbitrage-free term structure models, since they exclude a multitude of asymptotic maturity behavior. These severe implications serve as caution for modelers that not every specification is consistent with no arbitrage. Specifically, setting up a long yield, which decreases with positive probability or increases almost surely, imposes arbitrage opportunities.

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